Global stability of first-order methods with constant step size for coercive tame functions

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Abstract

We consider first-order methods for minimizing locally Lipschitz coercive functions that are tame in an o-minimal structure on the real field. We prove that if the method is approximated by subgradient trajectories, then the iterates eventually remain in a neighborhood of a connected component of the set of critical points. Under suitable regularity assumptions, this result applies to the subgradient method with momentum, the stochastic subgradient method with random reshuffling and momentum, and the random-permutations cyclic coordinate descent method.

Key words: differential inclusions, Kurdyka-Łojasiewicz inequality, semi-algebraic geometry

1 Introduction

Consider the unconstrained minimization problem

$$\inf_{x\in\mathbb{R}^n} f(x) := \frac{1}{N}\sum_{i=1}^{N} f_i(x) \tag{1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz for $i = 1, 2, \ldots, N$. Such unconstrained optimization problems are central in machine learning applications such as empirical risk minimization [16], low-rank matrix recovery [35, 38, 63], and the training of deep neural networks [33]. We study some widely used first-order methods, namely the subgradient method with momentum (Algorithm 1), the stochastic subgradient method with random reshuffling and momentum (Algorithm 2), and the random-permutations cyclic coordinate descent method (Algorithm 3). While implemented

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by machine learning practitioners [53, 1, 48], the analysis of these methods with constant step sizes seems to be absent from the literature when the objective is neither convex nor differentiable with a locally Lipschitz gradient (see Section 2).

In this paper, we provide global stability guarantees for first-order methods with constant step size for objective functions that are locally Lipschitz, coercive, and tame in an o-minimal structure on the real field (Definition 4). In order to do so, we show that the function values and the iterates of an iterative method eventually stabilize around some critical value (Theorem 1) and the set of critical points (Corollary 1) respectively, given that the method is approximated by subgradient trajectories of the objective function (Definition 3). As it turns out, all of the aforementioned first-order methods are approximated by subgradient trajectories of locally Lipschitz functions under method-dependent regularity assumptions (Propositions 1 and 2) as summarized in Table 1 (random reshuffling with momentum is short for stochastic subgradient method with random reshuffling and momentum). Therefore, these methods fit into our framework and their stability is guaranteed by Theorem 1 and Corollary 1. To the best of our knowledge, these methods have not been studied before at such generality as in this paper. In particular, we do not require the objective function to be convex and we do not require it to be differentiable with a locally Lipschitz gradient.

The function class studied in this paper is well-suited for applications. Indeed, seemingly all continuous objective functions of interest nowadays are locally Lipshitz and tame in an o-minimal structure on the real field, including all those appearing in the statistical learning textbook [23] by Friedman, Hastie, and Tibshirani. Many objective functions arising in data science applications are coercive due to the use of regularizers. Some objectives are naturally coercive, such as in symmetric low-rank matrix recovery problems [24, 35]. For functions that are not coercive, our results can still be applied if the iterates are uniformly bounded. We discuss this extension in Remark 2.

Our results rely on the connection between the iterates of first-order methods and the subgradient trajectories of the objective function. The subgradient trajectories of a locally Lipschitz function are solutions to a differential inclusion (i.e., equation (2) with $c = 1$). Previous works used the theory of differential inclusions [3] to study the stochastic subgradient method. Most of them are in the setting where a stochastic subgradient oracle is available (i.e., it generates a subgradient of the objective function in expectation) and the step sizes are diminishing. This differs from the assumptions of random reshuffling/permutation, which include sampling without replacement as a special case, and the constant step size in this paper. Under the former set of assumptions, the iterates of stochastic subgradient method converge almost surely to an internally chain transitive set of a differential inclusion [6, Theorem 3.6] [15, Corollary 4], with the proviso that the iterates are bounded almost surely and that the step sizes are not summable, among other assumptions. By additionally assuming that the objective function is Whitney stratifiable, the

Table 1: Standing assumption: f is coercive and tame.

<table>
<tr><th>Algorithm</th><th>Our assumption</th><th>Literature</th><th>Conclusion</th></tr>
<tr><td>Subgradient method with momentum</td><td>f locally Lipschitz</td><td>f differentiable with locally Lipschitz gradient [62, 46]</td><td rowspan="3">$f(x_k)$ and x_k eventually stay arbitrarily close to a critical value and a connected component of the set of critical points respectively, for all initial points in a bounded set X_0 and sufficiently small step sizes α</td></tr>
<tr><td>Random reshuffling with momentum</td><td>f_i locally Lipschitz and subdifferentially regular</td><td>no results</td></tr>
<tr><td>Random-permutations cyclic coordinate descent method</td><td>f continuously differentiable</td><td>f differentiable with locally Lipschitz gradient [4]</td></tr>
</table>

iterates subsequentially converge to critical points and the function values converge to a critical value almost surely [18, Corollary 5.9]. The recent work of Bianchi *et al* [11] used differential inclusions to analyze the stochastic subgradient method with constant step size. When a stochastic subgradient oracle is available along with other assumptions involving a Markov kernel, the iterates of the stochastic subgradient method eventually lie in the neighborhood of the critical points with high probability [11, Theorem 3], given that the constant step size is sufficiently small. The stochastic subgradient method with random reshuffling and diminishing step sizes was analyzed via differential inclusions recently in the work of Pauwels [49]. For Lipschitz continuous objectives, the iterates converge subsequentially to the set of critical points with respect to a conservative field [49, Corollary 6] assuming that they are bounded.

We next give the update rules of the first-order methods considered in this paper. Algorithm 1 is a generalization the framework proposed in the work of Kovachki and Stuart [30, (7)] from differentiable functions to locally Lipschitz functions. We denote by ∂f the Clarke subdifferential [17] (see Definition 1) of a locally Lipschitz function f. Algorithm 1 reduces to the heavy ball method [51] when $\gamma = 0$ and to the Nesterov's accelerated subgradient method [42, equation (2.2.22)] when $\beta = \gamma$ respectively. It also includes the vanilla subgradient method as a special case when $\beta = \gamma = 0$. Algorithm 2 is an extension of Algorithm 1 which exploits the composite nature of the objective function (1). Its update is the same as Algorithm 1 except that each step concerns only one component f_i, which is chosen at a random order at every iteration (epoch). This is exactly how stochastic subgradient method with momentum is implemented in practice (see for e.g., documentations from TensorFlow[1], PyTorch[2] and scikit-learn[3]). Last, Algorithm 3 is the random-permutations cyclic coordinate descent method, where $\nabla_i f(x) := [\nabla f(x)]_i e_i$, $[\nabla f(x)]_i$ is the ith entry of $\nabla f(x)$, and e_i is the ith canonical base. Similar to Algorithm 2, Algorithm 3 chooses a permutation of all the coordinates at every iteration and cycles through them.

The paper is organized as follows. Section 2 contains a literature review on the first-order methods and stochastic approximations with constant step size. Section 3 contains the convergence results for iterative methods that are approximated by subgradient trajectories. Finally, Section 4 explains how the first-order methods fit into the abstract framework of Section 3.

Algorithm 1 Subgradient method with momentum

choose step size $\alpha > 0$, momentum parameters $\beta \in (-1, 1)$, $\gamma \in \mathbb{R}$, constant $\delta > 0$, $x_{-1}, x_0 \in \mathbb{R}^n$ with $\|x_{-1} - x_0\| \leqslant \delta\alpha$
for $k = 0, 1, \ldots$ **do**
  $y_k = x_k + \gamma(x_k - x_{k-1})$
  $x_{k+1} \in x_k + \beta(x_k - x_{k-1}) - \alpha\partial f(y_k)$
end for

Algorithm 2 Random reshuffling with momentum

choose step size $\alpha > 0$, momentum parameters $\beta \in (-1, 1)$, $\gamma \in \mathbb{R}$, constant $\delta > 0$, $x_{-1,N-1}, x_0 \in \mathbb{R}^n$ with $\|x_{-1,N-1} - x_0\| \leqslant \delta\alpha$
for $k = 0, 1, \ldots$ **do**
  $x_{k,0} = x_k$
  $x_{k,-1} = x_{k-1,N-1}$
  choose a permutation σ^k of $\{1, 2, \ldots, N\}$
  for $i = 1, 2, \ldots, N$ **do**
    $y_{k,i} = x_{k,i-1} + \gamma(x_{k,i-1} - x_{k,i-2})$
    $x_{k,i} \in x_{k,i-1} + \beta(x_{k,i-1} - x_{k,i-2}) - \alpha\partial f_{\sigma_i^k}(y_{k,i})$
  end for
  $x_{k+1} = x_{k,N}$
end for

2 Literature review

We first review the first-order methods that we study in this paper, with a focus on results with constant step size. Next, we review the literature on stochastic

[1]https://www.tensorflow.org/api_docs/python/tf/keras/optimizers/SGD
[2]https://pytorch.org/docs/stable/generated/torch.optim.SGD.html
[3]https://scikit-learn.org/stable/modules/sgd.html

Algorithm 3 Random-permutations cyclic coordinate descent method

choose $x_0 \in \mathbb{R}^n$, step size $\alpha > 0$
for $k = 0, 1, \ldots$ **do**
  choose a permutation σ^k of $\{1, 2, \ldots, n\}$
  $x_{k,0} = x_k$
  for $i = 1, 2, \ldots, n$ **do**
    $x_{k,i} = x_{k,i-1} - \alpha \nabla_{\sigma_i^k} f(x_{k,i-1})$
  end for
  $x_{k+1} = x_{k,n}$
end for

approximations with constant step size.

The idea of accelerating gradient methods using momentum was originated with the heavy ball method [51]. When the objective function is strongly convex and twice differentiable, the heavy ball method enjoys a faster local convergence compared with the gradient method [51, Theorem 9]. A modified version of the heavy ball method was proposed and analyzed by Nesterov in 1983 [43], known as the Nesterov's accelerated gradient method. If one relaxes the assumption to being convex and differentiable with Lipschitz gradient, and requires that the infimum $f^* := \inf_{x \in \mathbb{R}^n} f(x) > -\infty$ is attained, then the iterates $(x_k)_{k \in \mathbb{N}}$ of Nesterov's accelerated gradient method satisfy $f(x_k) - f^* \leqslant O(1/k^2)$ [43, Theorem 1]. This is an improvement over the vanilla gradient method, whose convergence rate is $O(1/k)$ under the same set of assumptions [42, Corollary 2.1.2]. Let $\alpha \in (0, 2(1-\beta)/L)$, $\beta \in [0, 1)$, and $\gamma = 0$, if f is lower bounded, then the gradients $\nabla f(x_k)$ converge to zero [62, Lemmas 1,2,3] for any initial points $x_{-1}, x_0 \in \mathbb{R}^n$. If in addition the function is coercive and satisfies the Kurdyka-Łojasiewicz inequality [31] at every point and $x_{-1} = x_0$, then the iterates have finite length [46, Theorem 4.9]. To the best of our knowledge, the analysis of subgradient method with momentum has remained uncharted territory for nonsmooth objectives.

We next review the incremental subgradient method and the stochastic subgradient method with random reshuffling. Incremental subgradient method is a special of the stochastic subgradient method with random reshuffling where the components are visited in a fixed order. The history of the incremental gradient method can be traced back to the Widrow-Hoff least mean squares method [60] for minimizing a finite sum of convex quadratics in 1960. It was pointed out later by Kohonen that with sufficiently small constant step sizes, the limit points of the iterates of the least mean squares method are close to a minimum of the objective function [29]. With diminishing step sizes that are not summable but square summable, the least mean squares method converges to a minimum of the problem [36]. For convex objectives, the iterates $(x_k)_{k \in \mathbb{N}}$ of the incremental subgradient method with constant step size α satisfy $\liminf_{k \to \infty} f(x_k) \leqslant f^* + C\alpha$ for some $C > 0$ [40, Proposition 2.1], assuming that $f^* := \inf_{x\in\mathbb{R}^n} f(x) > -\infty$ and the subgradients of the components are uniformly bounded. We refer the readers to the survey paper [9], the textbook [8], and references therein for a more detailed discussion on the subject.

We turn our attention to the stochastic subgradient method with random reshuffling. As a stochastic version of the incremental subgradient method, existing results on the incremental subgradient method are helpful for the analysis of stochastic subgradient method with random reshuffling (see for e.g., recent works on the two methods [25, 26]). If the objective function is strongly convex and differentiable with Lipschitz gradients among other assumptions, the iterates of stochastic gradient method with random reshuffling and constant step size eventually lie in a neighborhood of the minimizer [39, Theorem 1] and a neighborhood of the minimum [44, Theorem 1] respectively, both in expectation. By relaxing the strong convexity assumption to mere convexity, the function values evaluated at the average iterates $\hat{x}_k := (\sum_{l=0}^k x_l)/k$ eventually lie in a neighborhood of the minimum in expectation [39, Theorem 3] [44, Remark 1]. By further removing the convexity assumption, the minimum norm of the gradients eventually lie in a neighborhood of zero in expectation [39, Theorem 4] [49, Corollary 1, Corollary 3] (see also [44, Theorem 4] for a similar result). The long-term behavior of the iterates for nonconvex and nonsmooth objective functions has so far remained elusive.

Despite the empirical success of incorporating momentum into the incremental gradient method/stochastic gradient method with random reshuffling [53], the theoretical understanding of such methods is limited. So far, the only guarantees available are for modified versions [56, 55]. The work of Tran *et al* [56] last year studied a modified version of stochastic gradient method with random reshuffling and heavy ball. The momentum is constant within every iteration (epoch) and is equal to the average of the gradients evaluated in the previous epoch. With the modification, the norm of gradients of the average iterates $\hat{x}_k$ eventually lie in a neighborhood of zero in expectation [56, Corollary 1], under various assumptions [56, Assumption 1]. A modified stochastic gradient method with random reshuffling and Nesterov's momentum was studied recently [55]. The momentum is only applied at the level of the outer loop, at the end of each iteration (epoch). In this setting, the function values eventually lie a neighborhood of the minimum when the component functions are convex [55, Theorem 1], among other assumptions.

We now review coordinate descent methods. We refer the readers to the survey paper [61] by Wright in 2015 for a more extensive introduction on the subject. The idea of coordinate descent methods is to optimize with respect to one variable at a time. It was first studied under the framework of univariate relaxation [47, Section 14.6]. With exact line search and almost cyclic rule or Gauss-Southwell rule for cycling over the coordinates, the coordinate descent method converges linearly to a minimizer of a strongly convex objective that is twice differentiable [37, Theorem 2.1]. More recently, the global convergence of random coordinate descent method was established for convex objectives with Lipschitz continuous partial derivatives

[41]. In contrast to cyclic coordinate descent methods, random coordinate descent methods choose a coordinate randomly at each iteration instead of following a cycling rule. Similar to the stochastic subgradient method with random reshuffling, the random-permutations cyclic coordinate descent method considered in this work is easier to implement than the random coordinate descent method as it requires only sequential access of the data [27]. Using [4, Lemma 3.3, remark 3.2], the convergence of the random-permutations cyclic coordinate descent method can be deduced for coercive functions with locally Lipschitz gradients. The superior performance of the random-permutations cyclic coordinate descent method was observed in numerical experiments, and was supported by analysis for convex quadratic objectives [34, 27]. For objective functions without a locally Lipschitz gradient, the study of the method appears to be absent from the literature.

Last, we review the existing results on stochastic approximations of differential inclusions with constant step size. They have led to recent advances on an oracle-based stochastic subgradient method with constant step size [11]. Given differential equations with Lipschitz right-hand sides over a finite time horizon, Kurtz proposed a sequence of discrete time stochastic processes that approaches their solutions with a probability that goes to one [32, Theorem (4.7)] (see also [5, Proposition 3.1]). Over an infinite time horizon, the sequence of corresponding invariant measures concentrates around the Birkhoff center of the differential equations [5, Corollary 3.2]. Later, Roth and Sandholm [52] extended these results to differential inclusions with upper semicontinuous right-hand sides and compact supports, along with other assumptions. More recently, the work of Bianchi *et al* [10] studied stochastic approximation with constant step size under a different set of assumptions, relaxing the compact support assumption from the previous literature. We refer the readers to the textbooks on Markov processes and stochastic approximations [21, 7] for more references on the subject. We remark that although the methods studied in this paper can be stochastic, the aforementioned results cannot be applied. This is mainly due to the fact that random reshuffling does not fit into the Markovian framework. Also, our results hold deterministically while the results we review only hold with high probability.

3 Global stability of first-order methods

We refer to an iterative method with constant step size as a set-valued mapping $\mathcal{M} : \mathbb{R}^{(\mathbb{R}^n)} \times (0,\infty) \times 2^{(\mathbb{R}^n)} \times \mathbb{N} \rightrightarrows (\mathbb{R}^n)^{\mathbb{N}}$ which, to an objective function $f : \mathbb{R}^n \to \mathbb{R}$, and a constant step size $\alpha \in (0,\infty)$, a set $X_0 \subset \mathbb{R}^n$, and a natural number $\bar{k}$ associates a subset of sequences in $\mathbb{R}^n$ whose $\bar{k}$th term is contained in X_0.

We next introduce several definitions. Let $\|\cdot\|$ be the induced norm of an inner product $\langle \cdot,\cdot \rangle$ on $\mathbb{R}^n$. Given a subset of S of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, consider the distance of x to S defined by $d(x,S) := \inf\{\|x-y\| : y \in S\}$. Let $B(a,r)$ denote the closed ball of center $a \in \mathbb{R}^n$ and radius $r > 0$, and let $B(S,r) := \cup_{a\in S} B(a,r)$ where $S \subset \mathbb{R}^n$. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz if for all $a \in \mathbb{R}^n$, there exist $r > 0$ and $L > 0$ such that $\|f(x) - f(y)\| \leqslant L\|x - y\|$ for all $x, y \in B(a,r)$. We use $[f \leqslant \Delta] := \{x \in \mathbb{R}^n : f(x) \leqslant \Delta\}$ to denote a sublevel set of a function $f : \mathbb{R}^n \to \mathbb{R}$ where $\Delta \in \mathbb{R}$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is coercive if $\lim_{\|x\|\to\infty} f(x) = \infty$.

Definition 1. *[17, Chapter 2] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The Clarke subdifferential is the set-valued mapping $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined for all $x \in \mathbb{R}^n$ by $\partial f(x) := \{s \in \mathbb{R}^n : f^\circ(x,d) \geqslant \langle s, d\rangle, \forall d \in \mathbb{R}^n\}$ where*

$$f^\circ(x,d) := \limsup_{\substack{y \to x \\ t \searrow 0}} \frac{f(y+td) - f(y)}{t}.$$

We say that $x \in \mathbb{R}^n$ is critical if $0 \in \partial f(x)$, and that $v \in \mathbb{R}$ is a critical value if there exists $x \in \mathbb{R}^n$ such that $0 \in \partial f(x)$ and $v = f(x)$. If f is continuously differentiable, then $\partial f(x) = \{\nabla f(x)\}$ [17, 2.2.4 Proposition].

Definition 2. *[3, Definition 1 p. 12] Given some real numbers a and b such that $a < b$, a function $x(\cdot)$ defined from $[a,b]$ to $\mathbb{R}^n$ is absolutely continuous if for all $\epsilon > 0$, there exists $\delta > 0$ such that, for any finite collection of disjoint subintervals $[a_1, b_1], \ldots, [a_m, b_m]$ of $[a,b]$ such that $\sum_{i=1}^m b_i - a_i \leqslant \delta$, we have $\sum_{i=1}^m \|x(b_i) - x(a_i)\| \leqslant \epsilon$.*

By virtue of [45, Theorem 20.8], a function $x : [a,b] \to \mathbb{R}^n$ is absolutely continuous if and only if it is differentiable almost everywhere on (a,b), its derivative $x'(\cdot)$ is Lebesgue integrable, and $\forall t \in [a,b], x(t) - x(a) = \int_a^t x'(t)dt$.

Finally, $\lfloor t \rfloor$ denotes the floor of a real number t which is the unique integer such that $\lfloor t \rfloor \leqslant t < \lfloor t \rfloor + 1$. The following definition is new to the best of our knowledge.

Definition 3. *An iterative method $\mathcal{M}$ is approximated by subgradient trajectories of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ (up to a positive multiplicative constant) if there exists $c > 0$ such that for any compact sets $X_0, X_1 \subset \mathbb{R}^n$, there exists $T > 0$ such that for all $\epsilon > 0$, there exists $\alpha_0 > 0$ such that for all $\alpha \in (0, \alpha_0]$, $\bar{k} \in \mathbb{N}$, and $(x_k)_{k\in\mathbb{N}} \in \mathcal{M}(f, \alpha, X_0, \bar{k})$ for which $x_0, \ldots, x_{\bar{k}} \in X_1$, there exists an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ such that*

$$x'(t) \in -c\partial f(x(t)), \quad \textit{for almost every } t \in [0,T], \quad x(0) \in X_0, \tag{2}$$

and $\|x_k - x((k-\bar{k})\alpha)\| \leqslant \epsilon$ for $k = \bar{k}, \ldots, \bar{k} + \lfloor T/\alpha \rfloor$.

Remark 1. *In the next section, we show that Algorithms 1, 2, and 3 satisfy Definition 3 (see Propositions 1 and 2). In order to do so, we always use the same strategy which consists in taking sequences generated by a given method with smaller and* *smaller constant step size, and show that a subsequence of their linear interpolations converges uniformly to a subgradient trajectory up to a finite time.*

Several discretization methods of initial value problems with differential inclusions were studied in [54, 3, 17, 22] (see also a survey on the subject by Dontchev and Lempio [19]). Assume that the set-valued mapping underlying the differential inclusion is upper semicontinuous with nonempty compact convex values, such that the norm of their elements are upper bounded by a linear function of the norm of the argument. Then over any finite time horizon, a subsequence of linear interpolations of the Euler method with smaller and smaller step sizes converges uniformly to a solution to the initial value problem [19, Theorem 2.2]. If in addition the set-valued mapping is bounded, then a class of linear multistep methods has the same convergence property as above [54, p. 127, Theorem] (see also [19, Convergence Theorem 3.2]).

However, the above results cannot be directly applied in this paper in order to prove that Definition 3 is satisfied. First, Algorithms 2 and 3 do not assume access to the set-valued mapping ∂f, which was a common assumption in the literature. Second, we require the approximation to hold uniformly for any compact set of initial points while only initial value problems were considered.

The class of locally Lipschitz functions is too broad to obtain any meaningful results on the first-order methods. We thus consider functions that are tame in o-minimal structures. O-minimal structures (short for order-minimal) were originally considered by Van den Dries, Pillay and Steinhorn [57, 50]. They are founded on the observation that many properties of semi-algebraic sets can be deduced from a few simple axioms [58]. Recall that a subset A of $\mathbb{R}^n$ is semi-algebraic [12] if it is a finite union of basic semi-algebraic sets, which are of the form $\{x \in \mathbb{R}^n : p_i(x) = 0,\ i = 1, \ldots, k;\ p_i(x) > 0,\ i = k+1, \ldots, m\}$ where $p_1, \ldots, p_m \in \mathbb{R}[X_1, \ldots, X_n]$ (i.e., polynomials with real coefficients).

Definition 4. *[59, Definition p. 503-506] An o-minimal structure on the real field is a sequence $S = (S_k)_{k \in \mathbb{N}}$ such that for all $k \in \mathbb{N}$:*

1. *S_k is a boolean algebra of subsets of $\mathbb{R}^k$, with $\mathbb{R}^k \in S_k$;*
2. *S_k contains the diagonal $\{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_i = x_j\}$ for $1 \leqslant i < j \leqslant k$;*
3. *If $A \in S_k$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ belong to S_{k+1};*
4. *If $A \in S_{k+1}$ and $\pi : \mathbb{R}^{k+1} \to \mathbb{R}^k$ is the projection onto the first k coordinates, then $\pi(A) \in S_k$;*
5. *S_3 contains the graphs of addition and multiplication;*
6. *S_1 consists exactly of the finite unions of open intervals and singletons.*

Note that S_1 are the semi-algebraic subsets of $\mathbb{R}$ and by [59, 2.5 Examples (3)], S_k contains the semi-algebraic subsets of $\mathbb{R}^k$. A subset A of $\mathbb{R}^n$ is definable in an o-minimal structure $(S_k)_{k\in\mathbb{N}}$ if $A \in S_n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is definable in an o-minimal structure if its graph, that is to say $\{(x,t) \in \mathbb{R}^{n+1} : f(x) = t\}$, is definable in that structure. A set $C \subset \mathbb{R}^n$ is tame [28] in an o-minimal structure $(S_k)_{k\in\mathbb{N}}$ if

$$\forall x \in \mathbb{R}^n,\ \forall r > 0, \quad C \cap B(x,r) \in S_n.$$

and a function $f : \mathbb{R}^n \to \mathbb{R}$ is tame if its graph is tame.

With the above definitions, we are now ready to state two technical lemmas.

Lemma 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. Let $\Delta \in \mathbb{R}$ and let $L > 0$ be a Lipschitz constant of f in $[f \leqslant \Delta]$. For any $\epsilon' > 0$, $B([f \leqslant \Delta - \epsilon' L], \epsilon') \subset [f \leqslant \Delta]$.*

Proof. We show that $B(a,\epsilon') \subset [f \leqslant \Delta]$ for all $a \in [f \leqslant \Delta - \epsilon' L]$. Indeed, if $b \in B(a,\epsilon') \setminus [f \leqslant \Delta]$, then there exists c in the segment $[a,b)$ such that $f(c) = \Delta$ and $\epsilon' L = \Delta - (\Delta - \epsilon' L) \leqslant f(c) - f(a) \leqslant L\|c - a\| < \epsilon' L$. □

Lemma 2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz tame function and let $\mathcal{M}$ be an iterative method with constant step size. Let $X \subset \mathbb{R}^n$ and L be a Lipschitz constant of f on X. For all $T, \epsilon', \alpha, c > 0$, $\bar{k} \in \mathbb{N}$, $(x_k)_{k\in\mathbb{N}} \in \mathcal{M}(f, \alpha, \mathbb{R}^n, 0)$, and for any subgradient trajectory $x : [0,T] \to \mathbb{R}^n$ of cf such that $x([0,T]) \subset X$, $x_k \in X$, and $\|x_k - x(\alpha(k - \bar{k}))\| \leqslant \epsilon'$ for $k = \bar{k}, \ldots, \bar{k} + \lfloor T/\alpha \rfloor$, we have*

$$f(x_k) \leqslant f(x((k-\bar{k})\alpha)) + \epsilon' L \leqslant f(x_{\bar{k}}) - c\int_0^{(k-\bar{k})\alpha} d(0, \partial f(x(s)))^2 \, ds + 2\epsilon' L$$

for $k = \bar{k}, \ldots, \bar{k} + \lfloor T/\alpha \rfloor$.

Proof. For $k = \bar{k}, \ldots, \bar{k} + \lfloor T/\alpha \rfloor$, we have

$$f(x_k) \leqslant f(x((k-\bar{k})\alpha)) + \epsilon' L \tag{3a}$$

$$= f(x(0)) - (f(x(0)) - f(x((k-\bar{k})\alpha))) + \epsilon' L \tag{3b}$$

$$\leqslant f(x_0) - (f(x(0)) - f(x((k-\bar{k})\alpha))) + 2\epsilon' L \tag{3c}$$

$$= f(x_0) - c\int_0^{(k-\bar{k})\alpha} d(0, \partial f(x(s)))^2 \, ds + 2\epsilon' L. \tag{3d}$$

In (3a) and (3c), we invoke the Lipschitz constant L of f on $X \ni x((k-\bar{k})\alpha), x_k$. (3d) is a consequence of [18, Lemma 5.2, Theorem 5.8] (see also [20]). □

We now turn to our main results, namely Theorem 1 and Corollary 1, which we prove using Lemmas 1 and 2.

Theorem 1 (Stability of function values)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz coercive tame function and let $\mathcal{M}$ be an iterative method with constant step size that is approximated by subgradient trajectories of f. For any bounded set $X_0 \subset \mathbb{R}^n$ and $\epsilon > 0$, there exist $\alpha_0, \Delta > 0$ such that for all $(x_k)_{k\in\mathbb{N}} \in \mathcal{M}(f, (0, \alpha_0], X_0, 0)$, we have $f(x_k) \leqslant \Delta$ for all $k \in \mathbb{N}$ and there exist a critical value f^* of f and $k_0 \in \mathbb{N}$ such that $|f(x_k) - f^*| \leqslant \epsilon$ for all $k \geqslant k_0$.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz coercive tame function. Since f is tame and coercive, there exists $\Delta > 0$ such that $X_0 \subset [f \leqslant \Delta/2]$ and Δ is not a critical value of f. By the definable Morse-Sard theorem [13, Corollary 9], f has finitely many critical values $f_1 > \cdots > f_p$ in $[f \leqslant \Delta]$ (and it has at least one since f is coercive and continuous). Since f is coercive and continuous, the compact sublevel sets $[|f - f_i| \leqslant \epsilon]$, $i = 1, \ldots, p$, are pairwise disjoint after possible reducing ϵ, which we may do without loss of generality. We may also assume that $f_1 + 2\epsilon \leqslant \Delta$. According to the Kurdyka-Łojasiewicz inequality [13, Theorem 14] (see also [2, Theorem 4.1]) and the monotonicity theorem [58, (1.2) p. 43] [31, Lemma 2], there exist $\rho > 0$ and a strictly increasing concave continuous definable function $\psi : [0, \rho) \to [0, \infty)$ that is continuously differentiable on $(0, \rho)$ with $\psi(0) = 0$ such that $d(0, \partial f(x)) \geqslant 1/\psi'(|f(x) - f_i|)$ for all $x \in [|f - f_i| \leqslant \epsilon]$ whenever $0 < |f(x) - f_i| < \rho$ for $i = 1, \ldots, p$. Without loss of generality, assume $\epsilon < \rho$ so that $d(0, \partial f(x)) \geqslant 1/\psi'(|f(x) - f_i|)$ for all $x \in [|f - f_i| \leqslant \epsilon]$ such that $f(x) \neq f_i$.

Consider a Lipschitz constant $L \geqslant 1$ of f in $[f \leqslant \Delta]$ and the quantity

$$M := \inf\{d(0, \partial f(x)) : |f(x) - f_i| \geqslant \epsilon/2,\ i = 1, \ldots, p,\ f(x) \leqslant \Delta\} > 0. \tag{4}$$

Since $\mathcal{M}$ is approximated by subgradient trajectories of f, by Definition 3 there exist $c, T > 0$, and $\alpha_0 \in (0, T/2)$ such that such that for all $\alpha \in (0, \alpha_0]$, $\bar{k} \in \mathbb{N}$, and $(x_k)_{k\in\mathbb{N}} \in \mathcal{M}(f, \alpha, [f \leqslant \Delta/2], \bar{k})$ for which $x_0, \ldots, x_{\bar{k}} \in [f \leqslant \Delta]$, there exists a subgradient trajectory $x : [0, T] \to \mathbb{R}^n$ of f up to the multiplicative constant c for which $x(0) \in [f \leqslant \Delta/2]$ and $\|x_k - x(\alpha(k - \bar{k}))\| \leqslant \epsilon'$ for $k = \bar{k}, \ldots, \bar{k} + \lfloor T/\alpha \rfloor$ where

$$\epsilon' := \min\left\{\frac{\Delta}{4L}, \frac{cM^2T}{24L}, \frac{\epsilon}{8L}, \frac{cT}{2L\psi'(\epsilon/2)^2}\right\} > 0.$$

Since $[|f - f_1| \leqslant \epsilon], \ldots, [|f - f_p| \leqslant \epsilon]$ are compact, after possibly reducing T and α_0 the statement still holds if one replaces the initial set $[f \leqslant \Delta/2]$ by $[|f - f_1| \leqslant \epsilon], [|f - f_2| \leqslant \epsilon], \ldots,$ or $[|f - f_p| \leqslant \epsilon]$.

From now on, we fix a constant step size $\alpha \in (0, \alpha_0]$. Consider a sequence $(x_k)_{k\in\mathbb{N}} \in \mathcal{M}(f, \alpha, X_0, 0) \subset \mathcal{M}(f, \alpha, [f \leqslant \Delta/2], 0)$ along with an associated subgradient trajectory $x : [0, T] \to \mathbb{R}^n$ of f up to the multiplicative constant c for which $x(0) \in [f \leqslant \Delta/2] \subset [f \leqslant \Delta - \epsilon' L]$ and $\|x_k - x(\alpha(k - \bar{k}))\| \leqslant \epsilon'$ for $k = \bar{k}, \ldots, \bar{k} + K$ where $\bar{k} = 0$ and $K := \lfloor T/\alpha \rfloor$. By Lemmas 1 and 2, for $k = 0, \ldots, K$, we have $f(x_k) \leqslant f(x(k\alpha)) + \epsilon' L \leqslant f(x(0)) + \epsilon' L \leqslant \Delta/2 + \epsilon' L \leqslant \Delta$ and

$$f(x_k) \leqslant f(x_0) - c\int_0^{k\alpha} d(0, \partial f(x(s)))^2\, ds + 2\epsilon' L. \tag{5}$$

If $c\int_0^{K\alpha} d(0,\partial f(x(s)))^2\, ds \geqslant 3\epsilon' L$, then we have $f(x_K) \leqslant f(x_0) - 3\epsilon' L + 2\epsilon' L \leqslant \Delta/2$ so that we may apply Lemmas 1 and 2 again with $\bar{k} = K$. Since the continuous function f is bounded below on the compact set $[f \leqslant \Delta/2]$, this process with constant decrease can only be repeated finitely many times. Thus there exist $v \in \mathbb{N}$ and an absolutely continuous function (again denoted $x(\cdot)$) such that $f(x_k) \leqslant f(x_{vK}) - c\int_0^{(k-vK)\alpha} d(0,\partial f(x(s)))^2\, ds + 2\epsilon' L$ and $\|x_k - x(\alpha(k-vK))\| \leqslant \epsilon'$ for $k = vK,\ldots,(v+1)K$ where $c\int_0^{K\alpha} d(0,\partial f(x(s)))^2\, ds < 3\epsilon' L$. Hence there exists $t' \in [0, K\alpha]$ such that $d(0,\partial f(x(t')))^2 \leqslant 3\epsilon' L/(cK\alpha) \leqslant 3\epsilon' L/(cT/2) \leqslant M^2/4$, where we use the fact that $\epsilon' \leqslant cM^2T/(24L)$. Since $d(0,\partial f(x(t'))) \leqslant M/2$ and $f(x(t')) \leqslant \Delta$, by definition of M in (4) there exists $i \in \{1,\ldots,p\}$ such that $|f(x(t')) - f_i| < \epsilon/2$. We also have that $f(x(t')) \leqslant f(x(0)) \leqslant f(x_{vK}) + \epsilon' L \leqslant \Delta/2 + \epsilon' L$. Thus $f_i < f(x(t')) + \epsilon/2 \leqslant \Delta/2 + \epsilon' L + \epsilon/2 \leqslant \Delta/2 + 3\epsilon/8$. For $k' = vK,\ldots,(v+1)K$, we have

$$
\begin{aligned}
|f(x_{k'}) - f_i| \leqslant & |f(x_{k'}) - f(x(\alpha(k'-vK)))| + |f(x(\alpha(k'-vK))) - f(x(t'))| + && \text{(6a)}\\
& |f(x(t')) - f_i| && \text{(6b)}\\
\leqslant & L\|x_{k'} - x(\alpha(k'-vK))\| + |f(x(0)) - f(x(K\alpha))| + \epsilon/4 && \text{(6c)}\\
\leqslant & L\epsilon' + 3\epsilon' L + \epsilon/2 && \text{(6d)}\\
\leqslant & \epsilon/8 + 3\epsilon/8 + \epsilon/2 && \text{(6e)}\\
= & \epsilon. && \text{(6f)}
\end{aligned}
$$

Indeed, (6a) is due to the triangular inequality. We invoke the Lipschitz constant L of f on $[f \leqslant \Delta]$ in order to bound the first term in (6a). In order to bound the second term in (6a), we use the fact that the composition $f \circ x$ is decreasing and $0 \leqslant \alpha(k'-vK) \leqslant t' \leqslant K\alpha$. (6d) holds because $\|x_{k'} - x(\alpha(k'-vK))\| \leqslant \epsilon'$ and $|f(x(0)) - f(x(K\alpha))| = c\int_0^{K\alpha} d(0,\partial f(x(s)))^2\, ds < 3\epsilon' L$. (6e) is due to $\epsilon' \leqslant \epsilon/(8L)$.

We next show that $f(x_k) \leqslant f_i + \epsilon$ for all $k \geqslant k' := vK$. Without loss of generality, we assume that $k' = 0$ so that by (6) we have $f(x_k) \leqslant f_i + \epsilon$ for $k = 0,\ldots,K$. We prove that $f(x_{K+1}) \leqslant f_i + \epsilon$, hence $f(x_k) \leqslant f_i + \epsilon$ for all $k \geqslant k'$ by induction. We distinguish two cases. If $f(x_1) < f_i - \epsilon$, then $f(x_{K+1}) \leqslant f(x_1) + 2\epsilon' L < f_i - \epsilon + \epsilon/4 \leqslant f_i + \epsilon$, where the first inequality follows from $x_1 \in [f \leqslant f_i - \epsilon] \subset [f \leqslant \Delta/2 + 3\epsilon/8 - \epsilon] \subset [f \leqslant \Delta/2]$ and Lemmas 1 and 2. If $x_1 \in [|f - f_i| \leqslant \epsilon]$, then let $x : [0,T] \to \mathbb{R}^n$ be an associated subgradient trajectory of f up to the multiplicative constant c such that $\|x_k - x(\alpha(k-1))\| \leqslant \epsilon'$ for $k = 1,\ldots,K+1$ and $x(0) \in [|f - f_i| \leqslant \epsilon]$. Note that for any $t \in [0, K\alpha]$, $f(x(K\alpha)) \leqslant f(x(t)) \leqslant f(x(0)) \leqslant f_i + \epsilon \leqslant \Delta - \epsilon \leqslant \Delta - \epsilon' L$. By Lemma 2, we have that $f(x_{K+1}) \leqslant f(x(K\alpha)) + \epsilon' L < f_i + \epsilon/2 + \epsilon/8 \leqslant f_i + \epsilon$, as desired. Otherwise, we have $f(x(t)) \in [f_i + \epsilon/2, f_i + \epsilon]$ for all $t \in [0, K\alpha]$. By the Kurdyka-Łojasiewicz inequality, we have $d(0,\partial f(x(t))) \geqslant 1/\psi'(f(x(t)) - f_i) \geqslant 1/\psi'(\epsilon/2) > 0$.

According to [18, Lemma 5.2, Theorem 5.8] (see also [20]), it holds that

$$
\begin{aligned}
f(x(K\alpha)) - f_i &\leqslant f(x(0)) - f_i - c\int_0^{K\alpha} d(0, \partial f(x(s)))^2 \, ds && \text{(7a)} \\
&\leqslant f(x(0)) - f_i - cK\alpha/\psi'(\epsilon/2)^2 && \text{(7b)} \\
&\leqslant f(x(0)) - f_i - cT/(2\psi'(\epsilon/2)^2) && \text{(7c)} \\
&\leqslant \epsilon - cT/(2\psi'(\epsilon/2)^2). && \text{(7d)}
\end{aligned}
$$

Thus $f(x_{K+1}) - f_i \leqslant f(x(K\alpha)) - f_i + f(x_{K+1}) - f(x(K\alpha)) \leqslant \epsilon - cT/(2\psi'(\epsilon/2)^2) + \epsilon' L \leqslant \epsilon$, where we used the fact that $\epsilon' \leqslant (cT)/(2L\psi'(\epsilon/2)^2)$.

If $|f(x_k) - f_i| \leqslant \epsilon$ for all $k \geqslant k'$, then the conclusion of the theorem follows. Otherwise, there exists $\hat{k} \geqslant k'$ such that $f(x_{\hat{k}}) < f_i - \epsilon \leqslant \Delta/2 + 3\epsilon/8 - \epsilon \leqslant \Delta/2$. Following the same argument as in the paragraph below (5), there exists $v' \in \mathbb{N}$ and an absolutely continuous function (again denoted $x(\cdot)$) such that $f(x_k) \leqslant f(x_{v'K}) - c\int_0^{(k-\hat{k})\alpha} d(0, \partial f(x(s)))^2 \, ds + 2\epsilon' L$ and $\|x_k - x(\alpha(k - v'K))\| \leqslant \epsilon'$ for $k = \hat{k} + v'K, \ldots, \hat{k} + (v'+1)K$ where $c\int_0^{K\alpha} d(0, \partial f(x(s)))^2 \, ds < 3\epsilon' L$. As before, it follows that there exist $t'' \in [0, T]$ and $j \in \{1, 2, \ldots, p\}$ such that $|f(x(t'')) - f_j| \leqslant \epsilon/2$. Since $f(x(t'')) \leqslant f(x(0)) \leqslant f(x_{\hat{k}+v'K}) + \epsilon' L \leqslant f(x_{\hat{k}}) + 3\epsilon' L < f_i - \epsilon + 3\epsilon/8 = f_i - 5\epsilon/8$, it holds that $f_j < f_i$. Replicating (6a)-(6e), we get $|f(x_{k''}) - f_j| \leqslant \epsilon$ for $k'' = \hat{k} + v'K, \ldots, \hat{k} + (v'+1)K$. By the same argument as in the previous paragraph, we have $f(x_k) \leqslant f_j + \epsilon$ for all $k \geqslant k'' := \hat{k} + (v'+1)K$. Since f only has finitely many critical values, the conclusion of the theorem follows. □

Theorem 1 gives a "weak convergence" result, in the sense that the function values evaluated at the iterates eventually stabilize around some critical value. In fact, a "strong convergence" result regarding the distance between the iterates and the set of critical points can be obtained without any additional assumptions. This is the subject of the following corollary. Note that while Corollary 1 implies Theorem 1, it is not clear how to prove Corollary 1 without Theorem 1.

Corollary 1 (Stability of iterates)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz coercive tame function and let $\mathcal{M}$ be an iterative method with constant step size that is approximated by subgradient trajectories of f. For any bounded set $X_0 \subset \mathbb{R}^n$ and $\epsilon > 0$, there exists $\alpha_0 > 0$ such that for all $(x_k)_{k\in\mathbb{N}} \in \mathcal{M}(f, (0, \alpha_0], X_0, 0)$, there exist a connected component C of the set of critical points of f and $k_0 \in \mathbb{N}$ such that $d(x_k, C) \leqslant \epsilon$ for all $k \geqslant k_0$.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz coercive tame function and let $\mathcal{M}$ be an iterative method with constant step size that is approximated by subgradient trajectories of f. Let X_0 be a bounded subset of $\mathbb{R}^n$ and let $\epsilon > 0$. By Theorem 1, there exists $\alpha_1, \Delta > 0$ such that for all $(x_k)_{k\in\mathbb{N}} \in \mathcal{M}(f, (0, \alpha_1], X_0, 0)$, $f(x_k) \leqslant \Delta$ for all $k \in \mathbb{N}$. Let L denote a Lipschitz constant of f on the compact set $[f \leqslant \Delta]$ and consider the quantity

$$M := \inf\{d(0, \partial f(x)) : d(x, S) \geqslant \epsilon/2, f(x) \leqslant \Delta\} > 0, \tag{8}$$

where S is the set of critical points of f. Since $\mathcal{M}$ is approximated by subgradient trajectories of f, by Definition 3 there exist $c, T > 0$, and $\alpha_2 \in (0, \alpha_1]$ such that for all $\alpha \in (0, \alpha_2], \bar{k} \in \mathbb{N}$ and $(x_k)_{k\in\mathbb{N}} \in \mathcal{M}(f, \alpha, [f \leqslant \Delta], \bar{k})$ for which $x_0, \ldots, x_{\bar{k}} \in [f \leqslant \Delta]$, there exists a subgradient trajectory $x : [0, T] \to \mathbb{R}^n$ of f up to the multiplicative constant c for which $x(0) \in [f \leqslant \Delta]$ and $\|x_k - x((k - \bar{k})\alpha)\| \leqslant \epsilon'$ for $k = \bar{k}, \ldots, \bar{k} + \lfloor T/\alpha \rfloor$ where $\epsilon' := \min\{\epsilon/4, cM^2T/(16(1+L)), \epsilon^2/(32(1+L)cT)\}$. Again by Theorem 1 there exists $\alpha_3 \in (0, \alpha_2]$ such that for all $(x_k)_{k\in\mathbb{N}} \in \mathcal{M}(f, (0, \alpha_3], X_0, 0)$, there exist a critical value f^* of f and $k_0 \in \mathbb{N}$ such that $|f(x_k) - f^*| \leqslant \epsilon'$ for all $k \geqslant k_0$. Let $\alpha_0 := \min\{\alpha_3, \epsilon'/(2c(1 + L)), T/2\}$.

Let $\alpha \in (0, \alpha_0]$, $(x_k)_{k\in\mathbb{N}} \in \mathcal{M}(f, \alpha, X_0, 0)$, and fix a corresponding f^* and k_0. We fix some $k \geqslant k_0$ from now on and show that $d(x_k, S) \leqslant \epsilon$. Since $(x_{k'})_{k'\in\mathbb{N}} \in \mathcal{M}(f, \alpha, [f \leqslant \Delta], k)$ and $(x_{k'})_{k'\in\mathbb{N}} \subset [f \leqslant \Delta]$, there exists a subgradient trajectory $x : [0, T] \to \mathbb{R}^n$ of f up to the multiplicative constant c for which $x(0) \in [f \leqslant \Delta]$ and $\|x_{k'} - x(\alpha(k' - k))\| \leqslant \epsilon'$ for $k' = k, \ldots, k + K$ where $K := \lfloor T/\alpha \rfloor$. By Lemma 2, we have

$$c \int_0^{K\alpha} d(0, \partial f(x(s)))^2 \, ds \leqslant f(x_k) - f(x_{k+K}) + 2\epsilon' L \leqslant 2\epsilon'(1 + L). \tag{9}$$

Thus there exists $t' \in [0, K\alpha]$ such that $d(0, \partial f(x(t')))^2 \leqslant 2\epsilon'(1 + L)/(cK\alpha) \leqslant 2\epsilon'(1 + L)/(cT/2) \leqslant M^2/4$, where we use the fact that $\epsilon' \leqslant cM^2T/(16(1 + L))$. As $f(x(t')) \leqslant f(x(0)) \leqslant \Delta$, we have $d(x(t'), S) \leqslant \epsilon/2$. It now suffices to show that $\|x_k - x(t')\| \leqslant \epsilon/2$. Notice that $\|x_k - x(0)\| \leqslant \epsilon' \leqslant \epsilon/4$ and

$$\begin{aligned}
\|x(0) - x(t')\| &\leqslant \int_0^{t'} \|x'(s)\| \, ds && \text{(10a)} \\
&= \int_0^{t'} c \, d(0, \partial f(x(s))) \, ds && \text{(10b)} \\
&\leqslant \sqrt{\int_0^{t'} c \, ds} \sqrt{\int_0^{t'} c \, d(0, \partial f(x(s)))^2 \, ds} && \text{(10c)} \\
&\leqslant \sqrt{\int_0^{T} c \, ds} \sqrt{\int_0^{K\alpha} c \, d(0, \partial f(x(s)))^2 \, ds} && \text{(10d)} \\
&\leqslant \sqrt{cT} \sqrt{2\epsilon'(1 + L)} && \text{(10e)} \\
&\leqslant \epsilon/4. && \text{(10f)}
\end{aligned}$$

Indeed, (10a) is due to triangular inequality. (10b) is a consequence of (2) and [18, Lemma 5.2, Theorem 5.8] (see also [20]). (10c) is due to the Cauchy-Schwarz inequality. (10f) is due $\epsilon' \leqslant \epsilon^2/(32(1+L)cT)$. Summing up, we have $|d(x_k,S) - d(x(t'),S)| \leqslant \|x_k - x(t')\| \leqslant \|x_k - x(0)\| + \|x(0) - x(t')\| \leqslant \epsilon/2$ and thus $d(x_k,S) \leqslant \epsilon$.

We have just shown that for all $d(x_k,S) \leqslant \epsilon$ for all $k \geqslant k_0$. Since $x_k \in [f \leqslant \Delta]$, we have that $d(x_k,S') = d(x_k,S) \leqslant \epsilon$, where $S' := S \cap B([f \leqslant \Delta], 2\epsilon)$. By the cell decomposition theorem [58, (2.11) p. 52], the definable compact set S' has finitely many compact connected components $C_1,\ldots,C_q$. Thus for each $k \geqslant k_0$, there exists $i_k \in \{1,\ldots,q\}$ such that $d(x_k,C_{i_k}) \leqslant \epsilon$. We next show that $d(x_{k+1},C_{i_k}) \leqslant \epsilon$, so that i_k can actually be chosen independently of k. Naturally, we have $d(C_i,C_j) := \inf\{\|x-y\| : (x,y) \in C_i \times C_j\} > 0$ for all $i \neq j$, otherwise $C_i \cap C_j \neq \emptyset$. Without loss of generality, we may assume that $\epsilon \leqslant \min\{d(C_i,C_j) : i \neq j\}/4$. It follows that, for all $j \neq i_k$, we have $d(x_k,C_j) \geqslant d(C_{i_k},C_j) - d(x_k,C_{i_k}) \geqslant 4\epsilon - \epsilon = 3\epsilon$. Similar to (10a)-(10f), we have $\|x(0) - x(\alpha)\| \leqslant \sqrt{c\alpha}\sqrt{2\epsilon'(1+L)} \leqslant \epsilon'$ since $\alpha \leqslant \alpha_0 \leqslant \epsilon'/(2c(1+L))$. Thus $\|x_{k+1} - x_k\| \leqslant \|x_{k+1} - x(\alpha)\| + \|x(\alpha) - x(0)\| + \|x(0) - x_k\| \leqslant 3\epsilon' \leqslant \epsilon$. Hence $d(x_{k+1},C_j) \geqslant d(x_k,C_j) - \|x_k - x_{k+1}\| \geqslant 3\epsilon - \epsilon = 2\epsilon$ for all $j \neq i_k$. Since $d(x_{k+1},S) = \min\{d(x_{k+1},C_j) : j = 1,\ldots,q\} \leqslant \epsilon$, we conclude that $d(x_{k+1},C_{i_k}) \leqslant \epsilon$. □

Remark 2. *The assumption that f is coercive in Theorem 1 and Corollary 1 can be replaced by requiring the iterates to be uniformly bounded for all sufficiently small step sizes when initialized in X_0. Precisely, we can ask for there to exist $\bar{\alpha}, r > 0$ such that $\mathcal{M}(f,(0,\bar{\alpha}],X_0,0) \subset B(0,r)^{\mathbb{N}}$. Indeed, one can then apply our results to a coercive function f_r which coincides with f in $B(0,2r)$, namely $f_r(x) := f(P_{B(0,2r)}(x)) + d(x,B(0,2r))$ for all $x \in \mathbb{R}^n$. It is clear that f_r is definable and coercive. In order to show that f_r is Lipschitz continuous, it suffices to prove $g_r(x) := f(P_{B(0,2r)}(x))$ is locally Lipschitz. Letting $L > 0$ denote a Lipschitz constant of f in $B(0,2r)$, we have $\|f(x) - f(y)\| \leqslant L\|x-y\|$. Therefore, $\|g_r(x) - g_r(y)\| = \|f(P_{B(0,2r)}(x)) - f(P_{B(0,2r)}(y))\| \leqslant L\|P_{B(0,2r)}(x) - P_{B(0,2r)}(y)\| \leqslant L\|x-y\|$. The last inequality follows from the convexity of $B(0,2r)$.*

4 Approximation of first-order methods by subgradient trajectories

The theory we developed in the previous section provides a unified framework under which global stability of iterative methods with constant step sizes can be established. In this section, we show that all of the first-order methods that we mentioned in Section 1 are approximated by subgradient trajectories under appropriate assumptions on the objective functions. As a result, Theorem 1 and Corollary 1 can readily be applied to conclude global stability of those methods.

We need the following lemma in order to prove the approximation of random reshuffling with momentum.

Lemma 3. *Let $f_1, \ldots, f_N$ be locally Lipschitz, $X \subset \mathbb{R}^n$ be bounded, $\delta \geqslant 0$, $\beta \in (-1,1)$, and $\gamma \in \mathbb{R}$. There exist $\delta', \bar{\alpha} > 0$ such that for all $\alpha \in (0, \bar{\alpha}]$, $\bar{k} \in \mathbb{N}$, and sequence $(x_{k,i})_{(k,i) \in \mathbb{N} \times \{0,\ldots,N\}}$ generated by random reshuffling with momentum (Algorithm 2) for which $x_0, \ldots, x_{\bar{k}} \in X$, we have*

$$\|x_{k,i} - x_{k,i-1}\| \leqslant \delta' \alpha$$

for $k = 0, \ldots, \bar{k}$ and $i = 0, \ldots, N$.

Proof. Let $r > 0$ such that $x_{-1,0} \in B(0, r/2)$ and $X \subset B(0, r/2)$. Since $f_1, f_2, \ldots, f_N$ are locally Lipschitz, their corresponding Clarke subdifferentials $\partial f_1, \partial f_2, \ldots, \partial f_N$ are upper semicontinuous [17, 2.1.5 Proposition (d)] with compact values [17, 2.1.2 Proposition (a)]. Thus, by [3, Proposition 3 p. 42] there exists $r' > \delta$ such that $\cup_{i=1}^N \partial f_i(B(0,r)) \subset B(0, r')$. Let

$$\delta' := \frac{r'}{1 - |\beta|} \quad \text{and} \quad \bar{\alpha} := \frac{r}{2\delta'(N + |\gamma|)}.$$

Fix any $\alpha \in (0, \bar{\alpha}]$, $\bar{k} \in \mathbb{N}$, and sequence $(x_{k,i})_{(k,i) \in \mathbb{N} \times \{0,\ldots,N\}}$ generated by random reshuffling with momentum (Algorithm 2) for which $x_0, \ldots, x_{\bar{k}} \in X$. We will prove the lemma using induction on (k, i) with the total order $\preccurlyeq$ defined by $(k_1, i_1) \preccurlyeq (k_2, i_2)$ if $k_1 < k_2$ or $k_1 = k_2$ and $i_1 \leqslant i_2$. For the base case, note that $\|x_{0,0} - x_{0,-1}\| \leqslant \delta\alpha < r'\alpha \leqslant \delta'\alpha$. Now fix any $(k, i) \in \{0, \ldots \bar{k}\} \times \{0, \ldots, N\}$ and assume that $\|x_{k',i'} - x_{k',i'-1}\| \leqslant \delta'\alpha$ for all $(k', i') \preccurlyeq (k, i-1)$ (we identify $(k'-1, N-1)$ with $(k', -1)$ for notational simplicity; possibly negative indices i are treated similarly throughout the paper). Then

$$\begin{aligned}
\|y_{k,i}\| &\leqslant \|y_{k,i} - x_{k,i-1}\| + \|x_{k,i-1} - x_{k,0}\| + \|x_{k,0}\| && \text{(12a)}\\
&\leqslant |\gamma| \|x_{k,i-1} - x_{k,i-2}\| + \sum_{j=1}^{i-1} \|x_{k,j} - x_{k,j-1}\| + \|x_k\| && \text{(12b)}\\
&\leqslant |\gamma|\delta'\alpha + (i-1)\delta'\alpha + \frac{r}{2} && \text{(12c)}\\
&\leqslant (|\gamma| + N - 1)\delta'\alpha + \frac{r}{2} && \text{(12d)}\\
&\leqslant r. && \text{(12e)}
\end{aligned}$$

Above, we use the triangular inequality in (12a). We apply the update rule and again the triangular inequality to obtain (12b). (12c) is a result of the inductive hypothesis. (12d) and (12e) follow from $i \leqslant N$ and $\alpha \leqslant \bar{\alpha} := r/(2\delta'(|\gamma| + N - 1))$ respectively.

Thus $x_{k,i} - x_{k,i-1} - \beta(x_{k,i-1} - x_{k,i-2}) \in -\alpha \partial f_{\sigma_i^k}(y_{k,i}) \subset \alpha B(0, r')$. Therefore,

$$\|x_{k,i} - x_{k,i-1}\| \leqslant |\beta| \|x_{k,i-1} - x_{k,i-2}\| + r'\alpha$$

$$\leqslant |\beta|\delta'\alpha + r'\alpha$$
$$= \delta'\alpha.$$

□

Recall that a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is subdifferentially regular [17, 2.3.4 Definition] if its generalized directional derivative agrees with the classical directional derivative, that is to say, we have

$$\limsup_{\substack{y \to x \\ t \searrow 0}} \frac{f(y+th) - f(y)}{t} \;=\; \lim_{t \searrow 0} \frac{f(x+th) - f(x)}{t}$$

for all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$, and the limit on the right hand side exists. We assume subdifferential regularity in Propositon 1 in order to guarantee that $\partial(f_1 + \cdots + f_N) = \partial f_1 + \cdots + \partial f_N$, while in general we only know that $\partial(f_1 + \cdots + f_N) \subset \partial f_1 + \cdots + \partial f_N$ holds [17, 2.3.3 Proposition] (see Remark 3). If we do not assume subdifferential regularity, Proposition 1 still holds with the same proof if in the conclusion we replace "approximated by subgradient trajectories of f" by "approximated by trajectories of the conservative field $(\partial f_1 + \cdots + \partial f_N)/N$ [14, Definition 1, Corollary 4] of f". Theorem 1 and Corollary 1 then hold with critical values and points associated with the conservative field under the additional assumption that $f_1, \ldots, f_N$ are definable [14, Theorems 5 and 6]. On the other hand, since Algorithm 1 does not consider the composite structure of the objective function, we do not require subdifferential regularity in order to obtain approximation and stability guarantees, as can be seen in Table 1.

Proposition 1. *Random reshuffling with momentum (Algorithm 2) is approximated by subgradient trajectories of composite functions $f = (f_1 + \cdots + f_N)/N$ up to the multiplicative constant $N/(1-\beta)$ where $f_1, \ldots, f_N$ are locally Lipschitz and subdifferentially regular.*

Proof. Let $\mathcal{M}$ denote the random reshuffling with fixed momentum parameters $\beta \in (-1,1)$, $\gamma \in \mathbb{R}$, and $\delta > 0$. Let $X_0, X_1 \subset \mathbb{R}^n$ be compact sets and consider $r > 0$ such that $X_0, X_1 \subset B(0, r/2) \subset \mathbb{R}^n$. By Lemma 3, there exist $\delta', \bar{\alpha} > 0$ such that for all $\alpha \in (0, \bar{\alpha}]$, $\tilde{k} \in \mathbb{N}$, and sequence $(x_{k,i})_{(k,i) \in \mathbb{N} \times \{0,\ldots,N\}}$ generated by random reshuffling with momentum (Algorithm 2) for which $x_0, \ldots, x_{\tilde{k}} \in B(0,r)$, we have that $\|x_{k,i} - x_{k,i-1}\| \leqslant \delta'\alpha$ for $k = 0, \ldots, \tilde{k}$ and $i = 0, \ldots, N$. Let $T := r/(4\delta' N \max\{1, |\gamma|\}) > 0$ and $\bar{k} \in \mathbb{N}$. We next show that any sequence $(x_{k,i})_{(k,i) \in \mathbb{N} \times \{0,\ldots,N\}}$ and $(y_{k,i})_{(k,i) \in \mathbb{N} \times \{1,\ldots,N\}}$ generated by Algorithm 2 with step size $\alpha \in (0, \min\{T/2, \bar{\alpha}\}]$ such that $x_0, \ldots, x_{\bar{k}} \in X_1$ and $x_{\bar{k}} \in X_0$ satisfy $x_{k,0}, \ldots, x_{k,N}, y_{k,1}, \ldots, y_{k,N} \in B(0,r)$ for $k = \bar{k}, \ldots, \bar{k} + K - 1$ where $K := \lfloor T/\alpha \rfloor + 1$.

Fix any such α and sequence generated by Algorithm 2. Note that $\alpha K = \alpha(\lfloor T/\alpha \rfloor + 1) \leqslant 2T$ and thus $\alpha \leqslant (2T)/K = r/(2K\delta' N \max\{1, |\gamma|\})$. As $x_0, \ldots, x_{\bar{k}} \in$

$B(0,r)$, we have that $\|x_{\bar{k},i}\| \leqslant \|x_{\bar{k}}\| + \sum_{j=1}^{i} \|x_{k,j} - x_{k,j-1}\| \leqslant r/2 + i\delta'\alpha \leqslant r/2 + ir/(2NK) \leqslant r/2 + r/(2K)$ for $i = 1, \ldots, N$, where we apply Lemma 3 with $\tilde{k} = \bar{k}$ in the second last inequality. In particular, $x_{\bar{k}+1} = x_{\bar{k},N} \in B(0, r/2 + r/(2K))$. Apply the previous argument recursively, we have that $x_{k,i} \in B(0, r/2 + (k-\bar{k})r/(2K) + ir/(2NK)) \subset B(0,r)$ for $k = \bar{k}, \ldots, \bar{k}+K-1$. By the update rule of Algorithm 2, $\|y_{k,i}\| \leqslant \|y_{k,i} - x_{k,i-1}\| + \|x_{k,i-1}\| = |\gamma|\|x_{k,i-1} - x_{k,i-2}\| + \|x_{k,i-1}\| \leqslant |\gamma|\delta'\alpha + r/2 + (k-\bar{k})r/(2K) + ir/(2NK) \leqslant r/(2NK) + r/2 + (k-\bar{k})r/(2K) + ir/(2NK) \leqslant r$ for $k = \bar{k}, \ldots, \bar{k}+K-1$ and $i = 1, \ldots, N$.

Let $(\alpha_m)_{m\in\mathbb{N}}$ be a positive sequence that converges to zero and let $(\bar{k}_m)_{m\in\mathbb{N}}$ be a sequence of natural numbers. For each $m \in \mathbb{N}$, we attribute a sequence of iterates $(x_k^m)_{k\in\mathbb{N}} \in \mathcal{M}(f, \alpha_m, X_0, \bar{k}_m)$ such that $x_0, \ldots, x_{\bar{k}} \in X_1$. We may assume that $\alpha_m \in (0, \min\{T/2, \bar{\alpha}\}]$ for any m, then $x_{k,0}^m, \ldots, x_{k,N}^m, y_{k,1}^m, \ldots, y_{k,N}^m \in B(0,r)$ for $k = \bar{k}_m, \ldots, \bar{k}_m + \lfloor T/\alpha_m \rfloor$. Consider the linear interpolation of the iterates $x_{\bar{k}_m}^m, x_{\bar{k}_m+1}^m, \ldots, x_{\bar{k}_m+\lfloor T/\alpha_m\rfloor+1}^m$, that is to say, the function $\bar{x}^m(\cdot)$ defined from $[0,T]$ to $\mathbb{R}^n$ by

$$\bar{x}^m(t) := x_k^m + (t - \alpha_m(k - \bar{k}_m))\frac{x_{k+1}^m - x_k^m}{\alpha_m}$$

for all $t \in [\alpha_m(k-\bar{k}_m), \min\{\alpha_m(k-\bar{k}_m+1), T\}]$ and $k \in \{\bar{k}_m, \ldots, \bar{k}_m + \lfloor T/\alpha_m \rfloor\}$. Since $B(0,r)$ is convex, it holds that $\|\bar{x}^m(t)\| \leqslant r$ for all $t \in [0,T]$. We also have $\|(\bar{x}^m)'(t)\| = \|(x_{k+1}^m - x_k^m)/\alpha_m\| \leqslant \sum_{i=1}^{N} \|x_{k,i}^m - x_{k,i-1}^m\|/\alpha_m \leqslant N\delta'$ for all $t \in [\alpha_m(k-\bar{k}_m), \min\{\alpha_m(k-\bar{k}_m+1), T\}]$ and $k \in \{\bar{k}_m, \ldots, \bar{k}_m + \lfloor T/\alpha_m \rfloor\}$. By successively applying the Arzelà-Ascoli and the Banach-Alaoglu theorems (see [3, Theorem 4 p. 13]), there exists a subsequence (again denoted by $(\alpha_m)_{m\in\mathbb{N}}$) and an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ such that $\bar{x}^m(\cdot)$ converges uniformly to $x(\cdot)$ and $(\bar{x}^m)'(\cdot)$ converges weakly to $x'(\cdot)$ in $L^1([0,T], \mathbb{R}^n)$. We next verify that the limit $x(\cdot)$ is a solution to the differential inclusion with initial condition

$$x'(t) \in -\frac{1}{1-\beta}\sum_{i=1}^{N} \partial f_i(x(t)), \quad \text{for almost every } t \in [0,T], \quad x(0) \in X_0. \tag{14}$$

By subdifferential regularity of $f_1, \ldots, f_N$, we have $\partial(\sum_{i=1}^{N} f_i) = \sum_{i=1}^{N} \partial f_i$ [17, p. 40, Corollary 3]. It is thus easy to see that such $x(\cdot)$ is a subgradient trajectory of f up to the multiplicative constant $c := N/(1-\beta) > 0$.

For any fixed $m \in \mathbb{N}$, we have that

$$x_{k,i}^m - x_{k,i-1}^m - \beta(x_{k,i-1}^m - x_{k,i-2}^m) \in -\alpha_m \partial f_{\sigma_i^k}(y_{k,i}^m) \tag{15}$$

for all $k \in \{\bar{k}_m, \ldots, \bar{k}_m + \lfloor T/\alpha_m \rfloor\}$ and $i \in \{0, \ldots, N\}$. For any fixed k, summing (15) up for $i = 1, \ldots, N$ yields

$$x_{k+1,0}^m - x_{k,0}^m - \beta(x_{k+1,-1}^m - x_{k,-1}^m) \in -\alpha_m \sum_{i=1}^{N} \partial f_{\sigma_i^k}(y_{k,i}^m).$$

Consider the linear interpolation of the iterates $x^m_{\bar{k}_m,-1}, x^m_{\bar{k}_m+1,-1}, \ldots, x^m_{\bar{k}_m+\lfloor T/\alpha_m\rfloor+1,-1}$, that is to say, the function $\bar{x}^m_{-1}(\cdot)$ defined from $[0,T]$ to $\mathbb{R}^n$ by

$$\bar{x}^m_{-1}(t) := x^m_{k,-1} + (t - \alpha_m(k - \bar{k}_m))\frac{x^m_{k+1,-1} - x^m_{k,-1}}{\alpha_m}$$

for all $t \in [\alpha_m(k-\bar{k}_m), \min\{\alpha_m(k-\bar{k}_m+1), T\}]$ and $k \in \{\bar{k}_m, \ldots, \bar{k}_m + \lfloor T/\alpha_m \rfloor\}$.

For almost every $t \in (0,T)$ and any neighborhood $\mathcal{N}$ of 0, there exists $m_0 \in \mathbb{N}$ such that for any $m \geqslant m_0$, there exists $k \in \{\bar{k}_m, \ldots, \bar{k}_m + \lfloor T/\alpha_m \rfloor\}$ such that

$$(\bar{x}^m)'(t) - \beta(\bar{x}^m_{-1})'(t) = \frac{x^m_{k+1,0} - x^m_{k,0}}{\alpha_m} - \beta\frac{x^m_{k+1,-1} - x^m_{k,-1}}{\alpha_m} \tag{16a}$$

$$\in -\sum_{i=1}^N \partial f_{\sigma_i^k}(y^m_{k,i}) \tag{16b}$$

$$\subset -\sum_{i=1}^N \left(\partial f_{\sigma_i^k}(x(t)) + \mathcal{N}/N\right) \tag{16c}$$

$$\subset -\sum_{i=1}^N \partial f_i(x(t)) + \mathcal{N}, \tag{16d}$$

where (16c) follows from upper semi-continuity of ∂f_i [17, 2.1.5 Proposition (d)] and

$$\begin{aligned}
\|y^m_{k,i} - x(t)\| &\leqslant \left\|y^m_{k,i} - \bar{x}^m(t)\right\| + \|\bar{x}^m(t) - x(t)\| \\
&= \left\|y^m_{k,i} - x^m_k - (t - \alpha_m(k-\bar{k}_m))\frac{x^m_{k+1} - x^m_k}{\alpha_m}\right\| + \|\bar{x}^m(t) - x(t)\| \\
&\leqslant \left\|y^m_{k,i} - x^m_k\right\| + \left\|x^m_{k+1} - x^m_k\right\| + \|\bar{x}^m(t) - x(t)\| \\
&\leqslant (|\gamma| + i)\delta'\alpha_m + N\delta'\alpha_m + \|\bar{x}^m(t) - x(t)\| \\
&\to 0
\end{aligned}$$

as $m \to \infty$. It remains to show that $(\bar{x}^m_{-1})'(\cdot)$ converges weakly to $x'(\cdot)$ in $L^1([0,T],\mathbb{R}^n)$. Indeed, by [3, Convergence Theorem p. 60], it then holds that $(x,(1-\beta)x') \in \operatorname{graph}\left(\sum_{i=1}^N \partial f_i\right)$ and thus (14) follows.

For any $m \in \mathbb{N}$ and almost every $s \in [0,T]$, $\|(\bar{x}^m_{-1})'(s)\| = \|(x^m_{k+1,-1} - x^m_{k,-1})/\alpha_m\| \leqslant \sum_{i=0}^{N-1} \|x^m_{k,i} - x^m_{k,i-1}\|/\alpha_m \leqslant N\delta'$ for some $k \in \mathbb{N}$. Thus it suffices to show that for all $t \in [0,T]$,

$$\int_0^t (\bar{x}^m_{-1})'(s)\; ds \to \int_0^t x'(s)\; ds.$$

Indeed, $\left\|\int_0^t (\bar{x}^m_{-1})'(s)\; ds - \int_0^t (\bar{x}^m)'(s)\; ds\right\| = \cdots$

$$= \left\|\bar{x}^m_{-1}(t) - \bar{x}^m_{-1}(0) - (\bar{x}^m(t) - \bar{x}^m(0))\right\|$$

$$
\begin{aligned}
&= \left\| x^m_{k,-1} + (t - \alpha_m(k-\bar{k}_m)) \frac{x^m_{k+1,-1} - x^m_{k,-1}}{\alpha_m} - x^m_{\bar{k}_m,-1} \right. \\
&\quad \left. - \left(x^m_{k,0} + (t - \alpha_m(k-\bar{k}_m)) \frac{x^m_{k+1,0} - x^m_{k,0}}{\alpha_m} - x^m_{\bar{k}_m,0} \right) \right\| \\
&\leqslant \left\| x^m_{k+1,-1} - x^m_{k+1,0} \right\| + \left\| x^m_{k,-1} - x^m_{k,0} \right\| + \left\| x^m_{\bar{k}_m,-1} - x^m_{\bar{k}_m,0} \right\| \\
&\leqslant \delta'\alpha_m + \delta'\alpha_m + \delta'\alpha_m \\
&\to 0,
\end{aligned}
$$

where $k = \bar{k}_m + \lfloor t/\alpha_m \rfloor$. As $x'(\cdot)$ is a weak limit of $(\bar{x}^m)'(\cdot)$, $(\bar{x}^m_{-1})'(\cdot)$ converges weakly to $x'(\cdot)$.

To sum up, we have shown that for every sequence $(\alpha_m)_{m\in\mathbb{N}}$ of positive numbers converging to zero and every sequence $(\bar{k}_m)_{m\in\mathbb{N}}$ of natural numbers, there exists a subsequence of natural numbers for which the corresponding linear interpolations uniformly converge towards a solution of the differential inclusion (14). The conclusion of the proposition now easily follows. To see why, one can reason by contradiction and assume that there exists $\epsilon > 0$ such that for all $\alpha_1 \in (0, \alpha_0]$, there exist $\alpha \in (0, \alpha_1]$, $\bar{k} \in \mathbb{N}$, and a sequence $(x^m_k)_{k\in\mathbb{N}} \in \mathcal{M}(f, \alpha_m, X_0, \bar{k})$ such that $x^m_0, \ldots, x^m_k \in X_1$, and for any solution $x(\cdot)$ to the differential inclusion (14), it holds that $\|x^m_k - x(\alpha_m(k-\bar{k}_m))\| > \epsilon$ for some $k \in \{\bar{k}_m, \bar{k}_m+1, \ldots, \bar{k}_m + \lfloor T/\alpha_m \rfloor\}$. We can then generate a sequence $(\alpha_m)_{m\in\mathbb{N}}$ of positive numbers converging to zero and a sequence $(\bar{k}_m)_{m\in\mathbb{N}}$ of natural numbers such that, for any solution $x(\cdot)$ to the differential inclusion (14), it holds that $\|x^m_k - x(\alpha_m(k-\bar{k}_m))\| > \epsilon$ for some $k \in \{\bar{k}_m, \bar{k}_m+1, \ldots, \bar{k}_m + \lfloor T/\alpha_m \rfloor\}$. Since there exists a subsequence $(\alpha_{\varphi(m)})_{m\in\mathbb{N}}$ such that $(\bar{x}^{\varphi(m)}(\cdot))_{m\in\mathbb{N}}$ uniformly converges to a solution to the differential inclusion (14), we obtain a contradiction. □

Remark 3. *To further see why subdifferential regularity is required for applying Definition 3 to Algorithm 2, consider a locally Lipschitz coercive semi-algebraic function* $f := (f_1 + f_2 + f_3)/3$ *where* $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ *are defined by*

$$
f_1(x) := \max\{x, 0\} \quad , \quad f_2(x) := \min\{x, 0\} \quad , \quad f_3(x) := x^2 \quad , \quad \forall x \in \mathbb{R}.
$$

Notice that f_1, f_2, f_3 *are all locally Lipschitz but* f_1 *is not subdifferentially regular. We have* $0 \in \partial f_1(0) = [0,1], 0 \in \partial f_2(0) = [1,0]$*, and* $0 \in \partial f_3(0) = \{0\}$*. Meanwhile* $0 \notin \partial f(0) = \{1/3\}$*. Thus random reshuffling with momentum can get stuck at* 0*. Therefore, the conclusion of Corollary 1 does not apply to this example.*

Proposition 2. *Random-permutations cyclic coordinate descent method (Algorithm 3) is approximated by subgradient trajectories of continuously differentiable functions* $f : \mathbb{R}^n \to \mathbb{R}$*.*

Proof. Similar to the proof of Proposition 1, let $X_0 \subset \mathbb{R}^n$ be a compact subset. Consider $r > 0$ such that $X_0 \subset B(0, r/2) \subset \mathbb{R}^n$. We would like to find $T > 0$

such that for any $\alpha \in (0, T/2], \bar{k} \in \mathbb{N}$, any sequence $(x_{k,i})_{(k,i)\in\mathbb{N}\times\{0,\ldots,n\}}$ generated by Algorithm 3 with step size α for which $x_{\bar{k},0} \in X_0$, we have that $x_{k,i} \in B(0,r)$ for all $k = \bar{k}, \ldots, \bar{k}+K-1$ and $i = 0, 1, \ldots, n$ where $K := \lfloor T/\alpha \rfloor + 1$. As ∇f is continuous, we have $M := \sup\{\|\nabla f(x)\| : x \in B(0,r)\} < \infty$. Let $T := r/(4Mn) > 0$. Then $\alpha K = \alpha(\lfloor T/\alpha \rfloor + 1) \leqslant 2T$ and $\alpha \leqslant 2T/K = r/(2KMn)$. For any $k = \bar{k}, \ldots, \bar{k}+K-1$ and $i = 0, \ldots, n-1$, if one assumes that $x_{k,i} \in B(0,r)$, then $\|x_{k,i+1} - x_{k,i}\| = \|\alpha \nabla_{\sigma_{i+1}^k} f(x_{k,i})\| \leqslant \alpha M \leqslant r/(2Kn)$. As $x_{\bar{k},0} \in B(0, r/2)$, by induction, we have $x_{k,i} \in B(0, r/2 + (kn+i)r/(2Kn)) \subset B(0,r)$ for any $k = \bar{k}, \ldots, \bar{k}+K-1$ and $i = 0, \ldots, n$.

We next show that Algorithm 3 is approximated by subgradient trajectories, which are solutions to the following differential equation with initial condition

$$x'(t) = -\nabla f(x(t)), \quad \text{for almost every } t \in [0,T], \quad x(0) \in X_0. \tag{19}$$

Denote by $\mathcal{M}$ the random-permutations cyclic coordinate descent method defined by Algorithm 3. Let $(\alpha_m)_{m\in\mathbb{N}}$ denote a sequence of positive numbers that converges to zero and $(\bar{k}_m)_{m\in\mathbb{N}}$ be a sequence of natural numbers. Without loss of generality, we may assume that $(\alpha_m)_{m\in\mathbb{N}} \subset (0, T/2]$. For each $m \in \mathbb{N}$, we attribute a sequence of iterates $(x_k^m)_{k\in\mathbb{N}} \in \mathcal{M}(f, \alpha_m, X_0, \bar{k}_m)$. Consider the linear interpolation of the iterates $x_{\bar{k}_m}^m, x_{\bar{k}_m+1}^m, \ldots, x_{\bar{k}_m+\lfloor T/\alpha_m \rfloor+1}^m$, that is to say, the function $\bar{x}^m(\cdot)$ defined from $[0,T]$ to $\mathbb{R}^n$ by

$$\bar{x}^m(t) := x_k^m + (t - \alpha_m(k - \bar{k}_m))\frac{x_{k+1}^m - x_k^m}{\alpha_m}$$

for all $t \in [\alpha_m k, \min\{\alpha_m(k+1), T\}]$ and $k \in \{\bar{k}_m, \ldots, \bar{k}_m + \lfloor T/\alpha_m \rfloor\}$. Recall that $x_k^m = x_{k,0}^m = x_{k-1,n}^m$. As we have shown in the first paragraph of the proof, $x_{k,i} \in B(0,r)$ for all $k = \bar{k}_m, \ldots, \bar{k}_m + \lfloor T/\alpha_m \rfloor$ and $i = 0, 1, \ldots, n$, thus $x_{\bar{k}_m}^m, x_{\bar{k}_m+1}^m, \ldots, x_{\bar{k}_m+\lfloor T/\alpha_m \rfloor+1}^m \in B(0,r)$. Since $B(0,r)$ is convex, it holds that $\|\bar{x}^m(t)\| \leqslant r$ for all $t \in [0,T]$. Observe that $(\bar{x}^m)'(t) = (x_{k+1}^m - x_k^m)/\alpha_m = -\sum_{i=1}^n \nabla_{\sigma_i^k} f(x_{k,i-1})$ for all $t \in (\alpha_m k, \min\{\alpha_m(k+1), T\})$ and $k \in \{\bar{k}_m, \ldots, \bar{k}_m + \lfloor T/\alpha_m \rfloor\}$. Hence, we have that $\|(\bar{x}^m)'(t)\| = \|\sum_{i=1}^n \nabla_{\sigma_i^k} f(x_{k,i-1})\| \leqslant \sum_{i=1}^n \|\nabla_{\sigma_i^k} f(x_{k,i-1})\| \leqslant nM$ for almost every $t \in [0,T]$. By successively applying the Arzelà-Ascoli and the Banach-Alaoglu theorems (see [3, Theorem 4 p. 13]), there exists a subsequence (again denoted $(\alpha_m)_{m\in\mathbb{N}}$) and an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ such that $\bar{x}^m(\cdot)$ converges uniformly to $x(\cdot)$ and $(\bar{x}^m)'(\cdot)$ converges weakly to $x'(\cdot)$ in $L^1([0,T], \mathbb{R}^n)$.

For almost every $t \in [0,T]$ and any α_m in the sequence, $t \in (\alpha_m k, \min\{\alpha_m(k+1), T\})$ for some $k \in \{\bar{k}_m, \ldots, \bar{k}_m + \lfloor T/\alpha_m \rfloor\}$. We fix any such t and any $\xi > 0$ from now on. As ∇f is continuous, there exists $\delta > 0$ such that $\|\nabla_i f(y) - \nabla_i f(x(t))\| \leqslant \|\nabla f(y) - \nabla f(x(t))\| \leqslant \xi/(2n)$, for all $y \in B(x(t), \delta)$ and $i = 1, \ldots, n$. Since $(\bar{x}^m(\cdot))_{m\in\mathbb{N}}$ converges uniformly to $x(\cdot)$, there exists $m_0 \in \mathbb{N}$ such that $\|\bar{x}^m(t) - x(t)\| \leqslant \min\{\epsilon/2, \delta/2\}$ for any $m \geqslant m_0$. As $\lim_{m\to\infty} \alpha_m = 0$, there exists $m_1 \geqslant m_0$ such that $\alpha_m \leqslant \delta/(4nM)$ for all $m \geqslant m_1$. We next show that $\|(\bar{x}^m)'(t) - \nabla f(x(t))\| \leqslant \xi/2$ for all $m \geqslant m_1$. Indeed, if it is the case, then

$$
\begin{aligned}
(\bar{x}_m(t), \bar{x}'_m(t)) \in B\left(x(t), \min\left\{\frac{\xi}{2}, \frac{\delta}{2}\right\}\right) \times B\left(-\nabla f(x(t)), \frac{\xi}{2}\right) \\
\subset \operatorname{graph}(-\nabla f) + B(0,\xi)
\end{aligned}
$$

and by [3, Convergence Theorem p. 60], it holds that $x'(t) = -\nabla f(x(t))$ for almost every $t \in [0,T]$. The sequence of initial iterates $(x^m(0))_{m\in\mathbb{N}}$ lies in the compact set X_0, hence its limit $x(0)$ lies in X_0 as well. As a result, $x(\cdot)$ is a solution to the differential inclusion (19).

Note that for any $i = 1,\ldots,n$,

$$
\begin{aligned}
\|x^m_{k,i-1} - \bar{x}^m(t)\| &= \left\| x^m_k - \alpha_m \sum_{j=1}^{i-1} \nabla_{\sigma^k_j} f(x^m_{k,j-1}) - x^m_k - (t - \alpha_m(k - \bar{k}_m))\frac{x^m_{k+1} - x^m_k}{\alpha_m} \right\| \\
&= \left\| \alpha_m \sum_{j=1}^{i-1} \nabla_{\sigma^k_j} f(x^m_{k,j-1}) + (t - \alpha_m(k - \bar{k}_m))\frac{x^m_{k+1} - x^m_k}{\alpha_m} \right\| \\
&\leqslant \alpha_m \left\| \sum_{j=1}^{i-1} \nabla_{\sigma^k_j} f(x^m_{k,j-1}) \right\| + \|x^m_{k+1} - x^m_k\| \\
&\leqslant \alpha_m \sum_{j=1}^{i-1} \left\| \nabla_{\sigma^k_j} f(x^m_{k,j-1}) \right\| + \alpha_m \|(\bar{x}^m)'(t)\| \\
&\leqslant \alpha_m (i-1) M + \alpha_m n M \\
&\leqslant 2\alpha_m n M \leqslant \delta/2.
\end{aligned}
$$

Thus, $\|x^m_{k,i-1} - x(t)\| \leqslant \|x^m_{k,i-1} - \bar{x}^m(t)\| + \|\bar{x}^m(t) - x(t)\| \leqslant \delta/2 + \min\{\xi/2, \delta/2\} \leqslant \delta$ for all $i = 1,\ldots,n$ and $m \geqslant m_1$. It follows that

$$
\begin{aligned}
\|(\bar{x}^m)'(t) - \nabla f(x(t))\| &= \left\| \sum_{i=1}^{n} \nabla_{\sigma^k_i} f(x^m_{k,i-1}) - \nabla f(x(t)) \right\| \\
&= \left\| \sum_{i=1}^{n} \Big(\nabla_{\sigma^k_i} f(x^m_{k,i-1}) - \nabla_{\sigma^k_i} f(x(t)) \Big) \right\| \\
&\leqslant \sum_{i=1}^{n} \left\| \nabla_{\sigma^k_i} f(x^m_{k,i-1}) - \nabla_{\sigma^k_i} f(x(t)) \right\| \\
&\leqslant \sum_{i=1}^{n} \frac{\xi}{2n} = \frac{\xi}{2}.
\end{aligned}
$$

To sum up, we have shown that for every sequence $(\alpha_m)_{m\in\mathbb{N}}$ of positive numbers converging to zero and every sequence $(\bar{k}_m)_{m\in\mathbb{N}}$ of natural numbers, there exists a subsequence of natural numbers for which the corresponding linear interpolations uniformly converge towards a solution of the differential equation (19). The conclusion of the proposition now easily follows using the same argument as the last paragraph in the proof of Proposition 1. □

Note that in the proof above, we do not make use of the set X_1 that appears in Definition 3. This is because that for any iterates $(x_k)_{k\in\mathbb{N}}$ generated by the random-permutations cyclic coordinate descent method, $(x_k)_{k\geqslant\bar{k}}$ is unrelated to $x_0,\ldots,x_{\bar{k}-1}$ if given $x_{\bar{k}}$.

Remark 4. *The assumption of being continuously differentiable in Proposition 2 cannot be replaced by being locally Lipschitz and subdifferentially regular. Consider the locally Lipschitz, subdifferentially regular, coercive, and semi-algebraic function* $f:\mathbb{R}^2\to\mathbb{R}$ *defined by* $f(x_1,x_2) := |x_1-4|+2|x_1x_2+1|+|x_2^2-1/4|$ *and* $x^*=(1,-1)$. *Note that* x^* *is not a critical point as*

$$0\notin\partial f(x^*)=\{(-1-\lambda,-1+\lambda):\lambda\in[-1,1]\}.$$

Meanwhile, we have that $\partial_1 f(x^*)=\partial_2 f(x^*)=[-2,0]\ni 0$. *Thus random-permutations cyclic coordinate descent method can get stuck at* x^*. *Therefore, the conclusion of Corollary 1 does not apply to this example.*

We conclude this paper by illustrating Theorem 1, Corollary 1, Proposition 1, and Proposition 2 on two examples. The first (Figure 1a) is nonsmooth and the second (Figure 1b) is continuously differentiable. One can see that the iterates indeed track a subgradient trajectory up to a certain time, then go on to track another subgradient trajectory, after which they stabilize around a critical point.

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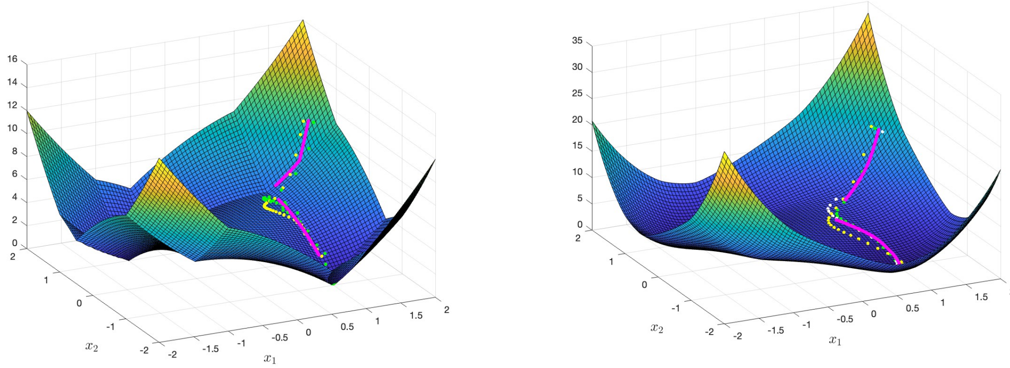

(a) $f(x_1, x_2) = |x_1^2 - 1| + 2|x_1 x_2 - 1| + |x_2^2 - 1|$.

(b) $f(x_1, x_2) = |x_1^2 - 1|^{3/2} + 2|x_1 x_2 - 1|^{3/2} + |x_2^2 - 1|^{3/2}$.

Figure 1: The subgradient method with momentum, random reshuffling with momentum, and random-permutations cyclic coordinate descent method are in yellow, green, and white respectively. Subgradient trajectories are in magenta.

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